

ON INJECTIVITY OF MAPS BETWEEN GROTHENDIECK GROUPS INDUCED BY COMPLETION

HAILONG DAO

ABSTRACT. We give an example of a local normal domain R such that the map of Grothendieck groups $G(R) \rightarrow G(\hat{R})$ is not injective. We also raise some questions about the kernel of that map.

1. INTRODUCTION

Let (R, m, k) be a local ring and \hat{R} the m -adic completion of R . Let $\mathcal{M}(R)$ be the category of finitely generated R -modules. The Grothendieck group of finitely generated modules over R is defined as:

$$G(R) = \frac{\bigoplus_{M \in \mathcal{M}(R)} \mathbb{Z}[M]}{\langle [M_2] - [M_1] - [M_3] \mid 0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0 \text{ is exact} \rangle}$$

In [KK], Kanoi and Kurano studied injectivity of the map $G(R) \rightarrow G(\hat{R})$ induced by flat base-change. They showed that such map is injective in the following cases : 1) R is Henselian, 2) R is the localization at the irrelevant ideal of a positively graded ring over a field, or, 3) R has only isolated singularity. Their results raise the question: Is the map between Grothendieck group induced by completion always injective?

In [Ho1], Hochster announced a counterexample to the above question:

Theorem 1.1. *Let k be a field. Let $R = k[x_1, x_2, y_1, y_2]_{(x_1, x_2, y_1, y_2)} / (x_1x_2 - y_1x_1^2 - y_2x_2^2)$. Let $P = (x_1, x_2)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \rightarrow G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.*

Hochster's example comes from the "direct summand hypersurface" in dimension 2 and is not normal. He predicted that there is also an example which is normal. The main purpose of this note is to provide such an example. We have:

Proposition 1.2. *Let $R = \mathbb{R}[x, y, z, w]_{(x, y, z, w)} / (x^2 + y^2 - (w+1)z^2)$. R is a normal domain. Let $P = (x, y, z)$ and $M = R/P$. Then $[M]$ is in the kernel of the map $G(R) \rightarrow G(\hat{R})$. However $[M] \neq 0$ in $G(R)$.*

This will be proved in Section 2. We note that Kurano and Srinivas has recently constructed an example of a local ring R such that the map $G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}$ is not injective (see [KS]). The ring in their example is not normal, and we do not know if a normal example exists in that context (i.e. with rational coefficients).

In section 3 we will discuss some questions on the kernel of the map $G(R) \rightarrow G(\hat{R})$.

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2. OUR EXAMPLE

We shall prove Proposition 1.2. First we need to recall some classical results:

Corollary 2.1. (*Swan, [Sw], Corollary 11.8*) *Let k be a field of characteristic not 2, $n > 1$ an integer and $R = k[x_1, \dots, x_n]/(f)$ where f is a non-degenerate quadratic form in $k[x_1, \dots, x_n]$. Then $G(R) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if $C_0(f)$, the even part of the Clifford algebra of f , is simple.*

Proposition 2.2. (*Samuel, see [Fo], Proposition 11.5*) *Let k be a field of characteristic not 2 and f be a non-degenerate quadratic form in $k[x_1, x_2, x_3]$. Let $R = k[x_1, x_2, x_3]/(f)$. If $f = 0$ has no non-trivial solution in k then $\text{Cl}(R) = 0$.*

Proposition 2.3. (*Kamoi-Kurano*) *Let $S = \bigoplus_{n \geq 0} S_n$ be a graded ring over a field S_0 and $S_+ = \bigoplus_{n > 0} S_n$. Let $A = S_{S_+}$. Then the map $G(S) \rightarrow G(A)$ induced by localization is an isomorphism.*

Proof. See the proof of Theorem 1.5 (ii) in [KK]. \square

Proposition 1.2 now follows from the following Propositions (clearly, R is normal, since the singular locus $V(x, y, z)$ has codimension 2):

Proposition 2.4. $[\hat{M}] = 0$ in $G(\hat{R})$.

Proof. $\hat{R} = \mathbb{R}[[x, y, z, w]]/(x^2 + y^2 - (w+1)z^2)$. We want to show that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. Let $\alpha = \sqrt{w+1}$ which is a unit in \hat{R} . Let $Q = (x, y - \alpha z)\hat{R}$. Then clearly Q is a height 1 prime in \hat{R} and $P\hat{R} = Q + (y + \alpha z)\hat{R}$. The short exact sequence:

$$0 \rightarrow \hat{R}/Q \rightarrow \hat{R}/Q \rightarrow \hat{R}/P\hat{R} \rightarrow 0$$

where the second map is the multiplication by $y + \alpha z$ shows that $[\hat{R}/P\hat{R}] = 0$ in $G(\hat{R})$. \square

Proposition 2.5. $[M] \neq 0$ in $G(R)$.

Proof. It is enough to show that $[M_P] \neq 0$ in $G(R_P)$. Let $K = \mathbb{R}(w)$ then $R_P \cong K[x, y, z]_{(x, y, z)}/(f)$ where $f = x^2 + y^2 - (w+1)z^2$. Let $S = K[x, y, z]/(f)$. Clearly f is a non-degenerate quadratic form. Since the rank of f is 3, an odd number, $C_0(f)$ is a simple algebra over K (see, for example, [La], Chapter 5, Theorem 2.4). By 2.1 and 2.3, $G(R_P) = G(S) = \mathbb{Z} \oplus \mathbb{Z}/(2)$. We claim that f has no non-trivial solution in K . Suppose it has. Then by clearing denominators, there are polynomials $a(w), b(w), c(w) \in \mathbb{R}[w]$ such that

$$a(w)^2 + b(w)^2 = (w+1)c(w)^2.$$

The degree of $a(w)^2 + b(w)^2$ is always even. The degree of $(w+1)c(w)^2$ is odd unless $c(w) = 0$. But then $a(w)^2 + b(w)^2 = 0$ which forces $a(w) = b(w) = 0$, a contradiction. By the claim and 2.2, $\text{Cl}(R_P) = \text{Cl}(S) = 0$. Thus $[R_P]$ and $[R_P/PR_P]$ generate $G(R_P) = \mathbb{Z} \oplus \mathbb{Z}/(2)$ (since the Grothendieck group is generated by $\{[R_P/Q], Q \in \text{Spec } R_P\}$ and $\dim R_P = 2$). Since $\mathbb{Z} \oplus \mathbb{Z}/(2)$ can not be generated by one element, $[R_P/PR_P]$ must be nonzero (it is easy to see that $[R_P/PR_P]$ is 2-torsion). \square

3. ON THE KERNEL OF THE MAP $G(R) \rightarrow G(\hat{R})$

In this section we raise some questions about the kernel of the map $G(R) \rightarrow G(\hat{R})$. First we fix some notations. Throughout this section we will assume, for simplicity, that R is excellent, and is a homomorphic image of a regular local ring T . Let $d = \dim R$. Let $A_i(R)$ be the i -th Chow group of R , i.e.,

$$A_i(R) = \frac{\bigoplus_{P \in \operatorname{Spec} R, \dim R/P = i} \mathbb{Z} \cdot [\operatorname{Spec} R/P]}{\langle \operatorname{div}(Q, x) \mid Q \in \operatorname{Spec} R, \dim R/Q = i+1, x \in R \setminus Q \rangle}$$

where

$$\operatorname{div}(Q, x) = \sum_{P \in \operatorname{Min}_R R/(Q, x)} l_{R_P}(R_P/(Q, x)R_P)[\operatorname{Spec} R/P].$$

For an abelian group A , we let $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. The Chow group of R is defined to be $A_*(R) = \bigoplus_{i=0}^d A_i(R)$. It is well known that there is a \mathbb{Q} -vector space isomorphism:

$$\tau_{R/T} : G(R)_{\mathbb{Q}} \rightarrow A_*(R)_{\mathbb{Q}}$$

(It is unknown whether this is independent of T). We also remark that the Grothendieck group $G(R)$ admits a filtration by $F_i G(R) = \langle [M] \in G(R) \mid \dim M \leq i \rangle$.

The existing examples on the failure of injectivity for the map $G(R) \rightarrow G(\hat{R})$ and the affirmative results in [KK] motivate the following question:

Question 3.1. *Suppose that R satisfies (R_n) (i.e., regular in codimension n). Then is $\ker(G(R) \rightarrow G(\hat{R}))$ contained in $F_{d-n-1} G(R)$?*

Question 3.1 is closely related to the following:

Question 3.2. *Suppose that R satisfies (R_n) . Then is the map $A_i(R) \rightarrow A_i(\hat{R})$ injective for $i \geq d - n$?*

In fact, if we allow rational coefficients, then the previous questions are equivalent. Let $G^i(R) = F_i G(R)/F_{i-1} G(R)$. Then clearly we have a decomposition:

$$G(R)_{\mathbb{Q}} = \bigoplus_{i=0}^d G^i(R)_{\mathbb{Q}}$$

Also, the Riemann-Roch map decomposes into isomorphisms $\tau^i : G^i(R)_{\mathbb{Q}} \rightarrow A_i(R)_{\mathbb{Q}}$, which make the following diagram:

$$\begin{array}{ccc} G^i(R) & \xrightarrow{\tau_{R/T}^i} & A_i(R) \\ \downarrow g_i & & \downarrow f_i \\ G^i(\hat{R}) & \xrightarrow{\tau_{\hat{R}/\hat{T}}^i} & A_i(\hat{R}) \end{array}$$

commutative. It follows that

$$\ker(G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}) \cong \bigoplus_i^d \ker(f_i) \cong \bigoplus_i^d \ker(g_i).$$

Thus we have:

Proposition 3.3. *Let R be an excellent local ring which is a homomorphic image of a regular local ring. Let $\dim R = d$ and let $0 < l \leq d$ be an integer. Then the maps $A_i(R)_{\mathbb{Q}} \rightarrow A_i(\hat{R})_{\mathbb{Q}}$ are injective for $i \geq l$ if and only if $\ker(G(R)_{\mathbb{Q}} \rightarrow G(\hat{R})_{\mathbb{Q}}) \subseteq F_{l-1} G(R)_{\mathbb{Q}}$.*

We do not know if 3.2 is true even if $l = 1$. Note that if R is normal, then both 3.1 and 3.2 are true for $l = 1$. In that situation $A_1(R) \cong \text{Cl}(R)$, and the map between class groups of R and \hat{R} is injective. Furthermore, it is well known that $G(R)/F_{d-2} G(R) \cong A_d(R) \oplus A_{d-1}(R)$ (see, for example [Ch], Corollary 1), so 3.1 is also true for $l = 1$.

Finally, one could formulate a stronger version of 3.1 as follows. Note that in both Hochster's example and the example presented here, the support of the modules given actually equal to the singular locus of R . So one could ask:

Question 3.4. *Let R be an excellent local ring. Let $X = \text{Spec } R$, $Y = \text{Sing}(X)$, $\hat{X} = \text{Spec } \hat{R}$ and $\hat{Y} = \text{Sing}(\hat{X})$. One then has a commutative diagram:*

$$\begin{array}{ccc} G(Y) & \xrightarrow{f} & G(X) \\ \downarrow & & \downarrow g \\ G(\hat{Y}) & \longrightarrow & G(\hat{X}) \end{array}$$

(Here $G(X)$ denotes the Grothendieck group of coherent \mathcal{O}_X -modules and the maps are naturally induced by closed immersions or flat morphisms). Is $\ker(g)$ contained in $\text{im}(f)$?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, SALT LAKE CITY, UT 84112-0090, USA

E-mail address: hdao@math.utah.edu